

# MULTILINE TRL REVEALED

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## **Abstract**

We reveal the techniques underlying actual implementation of the NIST MultiCal<sup>®</sup> software, an evolved, automated implementation of the Multiline TRL (Thru-Reflect-Line) calibration method for vector network analyzers (VNAs). We describe the sequence of events in MultiCal for the Multiline TRL calibration and show how the program operates more like a state-machine than a solver of simultaneous equations. Our report details the steps used in estimating the transmission-line propagation-constant and the VNA correction coefficients.

## **Introduction**

We reveal what is really going on in NIST's MultiCal<sup>®</sup> program in performing automated Multiline TRL (Thru-Reflect-Line) calibrations [1]<sup>1</sup>. It is written specifically for those readers already familiar with the Multiline method paper [1] and who desire to learn more. Although the original paper provides much detail, the actual implementation of the method in the NIST MultiCal software has not been publicly documented before now. We offer here the key features of the NIST code, providing important insight beyond the pure mathematics of the problem.

Writing Multiline TRL software logically start with the math, but software authors who are starting without the benefit of past experience are often confounded by spikes in their  $S$ -parameter sweeps and bothersome discontinuities in their propagation-constant data. This could understandably lead the author to a potentially erroneous conclusion that MultiCal must not be as good as other calibrations [3]. We must realize that reducing the Multiline method to practice in the form of MultiCal required the experience of many users and entailed an evolutionary process. Since the final code grew out of many fruitful discussions that took place at past ARFTG conferences, we take the opportunity of the 60th conference to reveal the numerical techniques

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<sup>1</sup> Though we document the Marks method [1], we note that the general TRL formulation of B. Bianco *et al.* [2] also encompasses the Multiline method.

and methods of MultiCal in hopes that it can assist other authors of Multiline calibration routines and sparks additional energetic debate leading to further improvement.

Since we cannot possibly include complete documentation of the code in a conference paper, we include what is possible and highlight here the key techniques and tests in MultiCal that are not so obvious. Our paper shows how the choice of using the Gauss-Markov estimator influences the selection of length difference pairs. We show how MultiCal works with an 8-term model when the network analyzers measure only three of the four wave variables at the same time, and we show how the roots of the analytic eigenvalue equations are assigned to the propagation terms and how the 8-term correction coefficients are estimated.

Various versions of MultiCal and its predecessor, *DEEMBED* have been distributed. This paper is specific to MultiCal Version 1.04a, though the principles are common to the more- and less-evolved versions of the code. The paper will not cover the LRM calibration and other capabilities of MultiCal.

### **Key Multiline Principles**

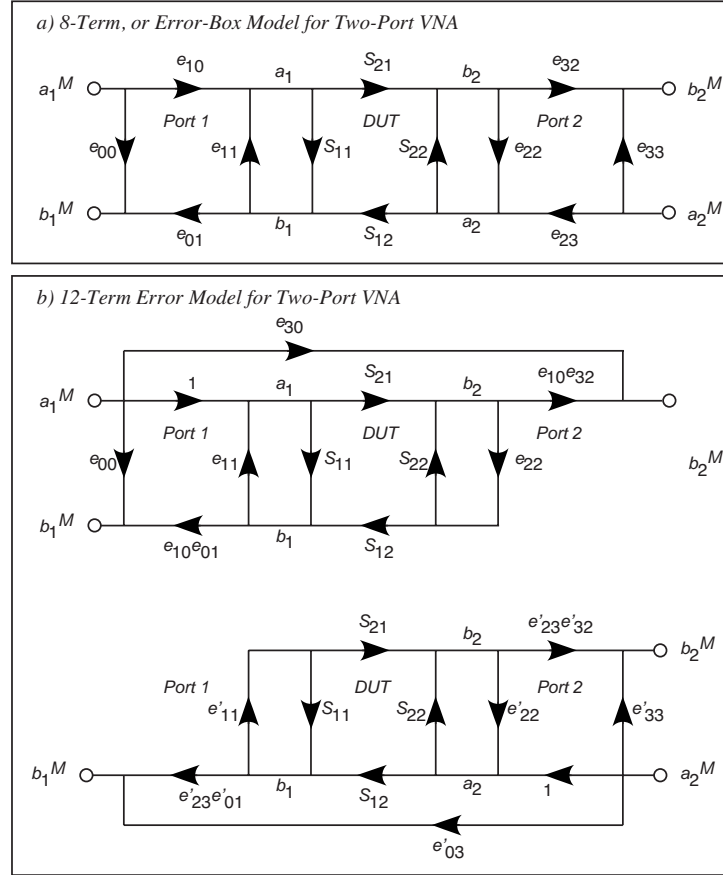
The Multiline method utilizes an ensemble of uncorrected two-port  $S$ -parameter measurements collected from a set of calibration artifacts, plus a measurement of the so-called “switch terms” [4] in order to compute the two-port VNA correction coefficients. The method defines transmission-line standards that differ only in length<sup>2</sup>, and an arbitrary reflection standard that is considered identical for both port connections. As a key part of this process, Multiline estimates the propagation-constant of the standards frequency-by-frequency, then computes the  $S$ -parameter correction coefficients in two parts, using the accurate estimate of the propagation-constant.

In examining MultiCal, we must keep in mind certain features of the Multiline method. First, the Multiline method is fundamentally an error-box formulation, as shown in Fig. 1a, and does not use independent forward and reverse sub-models as does the 12-term model as shown in Fig. 1b [5]. MultiCal handles differences between forward and reverse port match conditions by determining the switch-terms and correcting for these at the very beginning, and it can also apply an isolation term correction to the measured data. It then works exclusively in the error-box formulation. MultiCal will also compute the twelve terms for use with commercial VNAs, but it

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<sup>2</sup> In general, the standards are different lengths of waveguides, but for the purpose of this paper, we will consider only transmission-line standards. If hollow waveguides are to be used, the user enters a negative number for the real part of the effective relative permittivity.

determines the 12-term model out of its error-box model following previously described equivalencies [4].



**Figure 1** Two error models for two-port vector network analyzers: (a) 8-term model; (b) 12-term model with separate forward and reverse subcircuits.

Second, instead of applying numerical matrix solutions, the Multiline method and the MultiCal software analytically solve the eigenvalues of the measured cascade parameters (the  $ABCD$  version of the raw  $S$ -parameters). There are two eigenvalues for two-port systems, and MultiCal makes the best assignment of these roots to the transmission terms by testing for robustness in high-loss or low-loss regimes. This has not been described in detail before.

Next, we must realize that the Multiline formulation works on  $S$ -parameter correction relative to the characteristic impedance  $Z_0$  of the transmission-line standards (the line standards are defined as presenting a perfect match as the calibration reference plane). This means it works through the propagation-constant and correction coefficient problems without having to know the  $Z_0$  of the standards. If we stopped there, the method would produce corrected  $S$ -parameters that are defined only for waveguides like the standards themselves. Once  $Z_0$  of the standards is known,

MultiCal can transform the correction coefficients to any desired reference impedance, such as  $Z_{ref} = 50 + j0 \, \Omega$ .

Lastly, the Multiline method uses the available line standards and length differences to give better results than just averaging a number of repeated TRL calibrations. The Multiline method applies the Gauss–Markov theorem to form a “best” linear unbiased estimator (BLUE) for the propagation-constant *and* the error-box parameters. MultiCal ensures that the estimator will not encounter a nonsingular matrix by identifying one line standard as the common line at each frequency point, and forming line pairs with this common standard. The other length differences that are available are not used, only this select set. The following sections provide further explanation.

### ***Computation of the Propagation-constant***

The method requires an accurate determination of the propagation-constant  $\gamma$  of the transmission-line standards in order to compute error-box coefficients and to then translate the measurement reference plane, at the user’s request. By finding  $\gamma$  first, MultiCal demonstrates a robust method of characterizing transmission-lines without the need for a full VNA calibration. Conceptually, we prefer to isolate the computation of  $\gamma$  in the MultiCal code and to deal with it first. The computation follows seven steps:

1. Measure or import uncorrected  $S$ -parameters for available transmission-line standards.
2. Apply switch-term correction to uncorrected  $S$ -parameters if not previously accounted for by a first-tier VNA calibration.
3. Compute an estimate of the propagation-constant  $\gamma_{est}$ , based on user-supplied transmission-line parameters.
4. With  $\gamma_{est}$ , identify a common transmission-line to use in forming the line pairs.
5. Analytically solve the eigenvalues that will give actual  $e^{\pm\gamma\Delta\ell}$  values for select line pairs.
6. Determine correct eigenvalue to  $e^{\pm\gamma\Delta\ell}$  assignment using multiple selection criteria for both low- and high-loss transmission-line standards.
7. Compute a best estimate of  $\gamma$  and the equivalent representation of effective permittivity  $\epsilon_{eff}$ .

We provide descriptions for each.

#### *1. Measure uncorrected (raw) S-parameter data for available transmission-line standards.*

MultiCal acquires two-port  $S$ -parameters from commercial vector network analyzers or from data files. In both cases, it can accept totally uncorrected data and perform a first-tier calibration,

or it can receive data partially corrected for switch terms and perform a second-tier calibration. With the exception of switch term corrections (Step 2, below), the propagation-constant calculation is performed in the same way for both cases.

The user initially specifies multiple transmission-line standards with a physical length  $\ell_i$  for each, a THRU standard and its physical length<sup>3</sup>, a REFLECT standard with its type (open or short), and the location of the reflection plane relative to the connector plane (the probe pads for on-wafer connections). Though the REFLECT standard is not used in computing the propagation-constant, we need to compute the  $S$ -parameter correction coefficients; uncorrected data must be acquired for all standards in order for MultiCal to proceed with the computation.

Commonly, MultiCal works with frequency-swept data. The user defines a frequency sweep or frequency list in the VNA (or data file) and MultiCal acquires a matrix of measured  $S$ -parameters for each frequency point<sup>4</sup>. It completes this for each standard  $i$  specified in a user-supplied list.

$$\mathbf{S}^i = \begin{bmatrix} S_{11}^i & S_{12}^i \\ S_{21}^i & S_{22}^i \end{bmatrix} \quad (1)$$

Here  $\mathbf{S}^i$  is the matrix of measured (or imported) two-port  $S$ -parameters for standard  $i$  at any frequency point. MultiCal saves all these data in one three-dimensional array (2 x 2 x number of frequencies).

## 2. Apply switch-term correction to uncorrected $S$ -parameters, if necessary.

If switch-term correction is required (1st-Tier Calibration Mode) and MultiCal is acquiring data from a four-coupler VNA (Fig. 2), MultiCal measures the switch terms using the user parameters and then corrects the measurements of all standards.

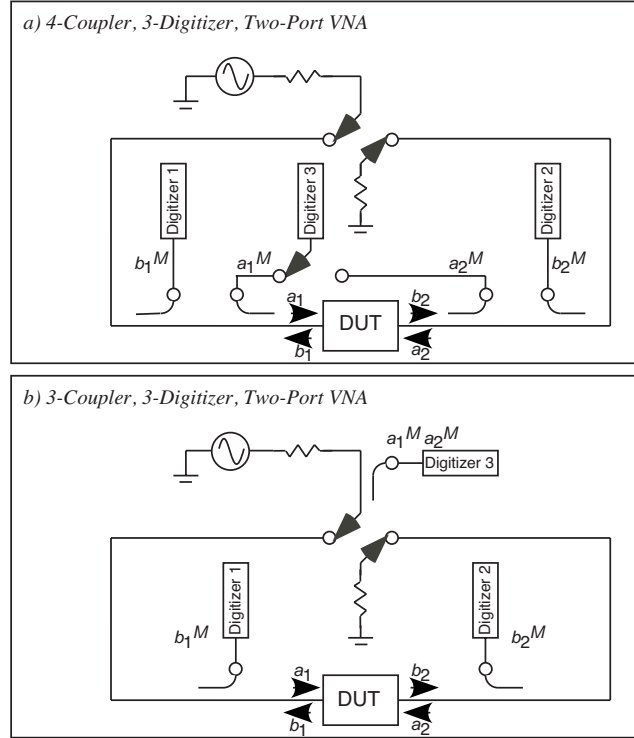
Presently, there are two common commercial VNA architectures, each distinguished by the number and location of directional couplers and digitizers, as shown in Fig. 2. In both of the cases, only three of the four wave-variables ( $a_1^M$ ,  $b_1^M$ , and  $b_2^M$ ; or  $a_2^M$ ,  $b_2^M$ , and  $b_1^M$ ) are measured (or used) at a time. When the source is switched from supplying a stimulus to Port 1 for forward measurements to supplying a stimulus at Port 2 for reverse measurements, a terminating load

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<sup>3</sup> MultiCal does not provide any assistance in choosing optimal line lengths. The choice of lengths should ensure that there is at least one line pair that gives a transmission coefficient phase difference other than 0° or 180° (or ideally, values that fall within 20° of these ill-conditioned points) at all frequencies of interest.

<sup>4</sup> The point spacing in the frequency list does not need to be uniform, although it seems to work best if the list is monotonically increasing in frequency, since MultiCal will revise the estimated effective permittivity value at a given frequency point, based on its estimate of the propagation-constant for the preceding frequency point.

(nominally  $50\ \Omega$ ) is switched from Port 2 to Port 1, respectively. Without measurements of all four wave-variables, there are in effect two different instrument states, and consequently switch-terms for the 8-term model, or error-box formulation [4,5]. These switch terms are naturally accounted for in forward and reverse subcircuits of the 12-term formulation and can be directly related to the switch-term corrected error-box formulation.



**Figure 2** Two-port VNA architectures: (a) VNA with four couplers but using only three digitizers simultaneously; (b) VNA with only three couplers and digitizers. The latter cannot measure the switch terms with MultiCal alone.

The user tells MultiCal whether the data to be used are uncorrected (1st-Tier Calibration Mode) or whether the data came from a VNA that already applied a correction (2nd-Tier Calibration Mode). In the 2nd-Tier Mode, MultiCal considers the switch terms to be accounted for and does nothing further. This is the only valid MultiCal mode when using a three-coupler VNA. In the 1st-Tier Mode, MultiCal will either make a measurement of the switch terms of the VNA, or look for a special data file (named “GTHRU”) that contains the switch-term information.

When controlling a VNA, MultiCal measures the uncorrected forward and reverse switch terms  $\Gamma_F$  and  $\Gamma_R$  by defining user parameters in the VNA and measuring these at the same time the user connects and measures the THRU standard:

$$\Gamma_F = \left. \frac{a_{2m}}{b_{2m}} \right|_{Source = Port1} \quad \text{and} \quad (2)$$

$$\Gamma_R = \left. \frac{a_{1m}}{b_{1m}} \right|_{Source = Port2}, \quad (3)$$

where  $a_{jm}$  and  $b_{im}$  are the uncorrected measured wave-variables. These are measured at each frequency in the list.

If instead, MultiCal is importing data from a file, it looks for the same two switch terms, using the data in the position normally reserved for  $S_{11}$  as  $\Gamma_R$ , and the data in the position normally reserved for  $S_{22}$  as  $\Gamma_F$ .

Now, if the network analyzer were a true four-coupler, four-digitizer type, such as the recently described large-signal network analyzers [6], it would simultaneously digitize and record  $a_{1m}$ ,  $b_{1m}$ ,  $a_{2m}$ , and  $b_{2m}$ . The  $S$ -parameters would then be uniquely defined for any termination placed beyond the couplers; in this case switch terms would not be an issue in formulating the model.

The user can also specify whether or not MultiCal should include an isolation correction for high-attenuation applications. These terms are the isolation coefficients of the 12-term model. Since MultiCal is working strictly with an 8-term model (that is, no cross-talk terms), it corrects the measured data as shown below and performs its computations in the 8-term formalism. It corrects the switch-terms first:

$$\Gamma_{IC,F} = \frac{\Gamma_F}{1 - ISO_F / S_{21}^{thru}} \quad \text{and} \quad (4)$$

$$\Gamma_{IC,R} = \frac{\Gamma_R}{1 - ISO_R / S_{12}^{thru}}, \quad (5)$$

where  $ISO_F$  and  $ISO_R$  are the uncorrected transmission parameters measured in the forward and reverse configuration when the ports are terminated with a nominally impedance-matched load.

MultiCal then isolation-corrects the transmission parameters of each standard:

$$S_{IC,12}^i = S_{12}^i - ISO_R \quad \text{and} \quad (6)$$

$$S_{IC,21}^i = S_{21}^i - ISO_F. \quad (7)$$

Finally, switch term-correction is applied to the resulting  $S$ -parameters of each standard:

$$\mathbf{S}_{SC}^i = \frac{1}{D^i} \begin{bmatrix} S_{11}^i - S_{IC,12}^i S_{IC,21}^i \Gamma_{IC,F} & S_{IC,12}^i - S_{11}^i S_{IC,12}^i \Gamma_{IC,R} \\ S_{IC,21}^i - S_{22}^i S_{IC,21}^i \Gamma_{IC,F} & S_{22}^i - S_{IC,12}^i S_{IC,21}^i \Gamma_{IC,R} \end{bmatrix}, \quad (8)$$

where

$$D^i = 1 - S_{IC,12}^i S_{IC,21}^i \Gamma_{IC,F} \Gamma_{IC,R}.$$

If isolation is omitted, only the switch-term correction is applied. With the data in this form, MultiCal uses the switch-term-corrected  $S$ -parameters to compute the propagation-constant and correction coefficients.

### 3. Compute estimate of propagation-constant $\gamma_{est}$

MultiCal requires users to specify an accurate estimate for  $\epsilon_{r,eff}$ , the effective complex relative permittivity of their transmission-line standards. Users can acquire this estimate with a field simulation of the line, which in certain cases may be a static field solution. For on-wafer structures, the conformal mapping approximations of Ref. [7] are often useful. The MultiCal user gets to enter only one value for the real part  $\epsilon'_{r,eff}$  without specifying a frequency. For lossy lines, the user may enter an estimate of the imaginary part of the effective relative permittivity  $\epsilon''_{r,eff}$  at 1 GHz, entering negative numbers to indicate loss. Using these values, MultiCal computes an estimated propagation-constant for the first frequency point in the list:

$$\gamma_{est} = j \frac{2\pi f}{100c} \sqrt{\epsilon'_{r,est} + j \frac{\epsilon''_{r,est}}{f/10^9}}. \quad (9)$$

Here,  $\gamma_{est}$  is represented in units per  $10^{-2}$  m,  $c$  is the speed of light in vacuum, and the frequency dependence of  $\epsilon''_{r,eff}$  is approximated using a  $1/f$  scaling, normalized to the user entry point. Equation (9) gets applied for only the *first* frequency point in the list. MultiCal reassigns  $\gamma_{est}$  at subsequent frequency points, using its most recent solution of  $\gamma$  to estimate the next point:

$$\gamma_{est}(f_{+1}) = \text{Re}\{\gamma(f)\} + j \text{Im}\{\gamma(f)\} \frac{f_{+1}}{f}, \quad (10)$$

where  $f_{+1}$  indicates the next frequency in the list.

### 4. Identify the optimal common line to use in forming line-length differences.



With the array of  $\gamma_{\text{est}}$  values, MutliCal identifies a common line to use at each frequency point. The common line is the one used to form the line-length differences in subsequent calculations and MultiCal may change this common line at each frequency point, as shown below.

This is a key point that authors of other Multiline software need to keep in mind. MultiCal does not simply use the THRU standard as the common line at all frequencies. Instead, it chooses a common line based on the resulting phase differences at each frequency point  $f$ . MutliCal follows this procedure:

- a) Start by taking the first standard  $L1$  (possibly the THRU standard) as the common line and estimate the effective phase differences  $\phi_{\text{eff}}$  between it and all the other line standards  $Lj$ , based on  $\gamma_{\text{est}}$  and the length differences  $\Delta l = l_{Lj} - l_{L1}$ :

$$\phi_{\text{eff}} = \sin^{-1} \left| \frac{e^{-\gamma_{\text{est}} \Delta l} - e^{+\gamma_{\text{est}} \Delta l}}{2} \right|, \quad (11)$$

where this definition of  $\phi_{\text{eff}}$  is used since it is more robust for both low- and high-loss transmission-lines and takes values between 0 and 90°. MultiCal sets  $\phi_{\text{eff}}$  to 0 if the argument of the arcsine happens to exceed 1 due to measurement noise.

- b) Record the minimum  $\phi_{\text{eff}}$  in the set when using  $L1$  as the common line at this frequency. This represents the line pair that would give the largest calibration error at this frequency.

Repeat steps a) and b), using in turn all other lines as the common line; from  $L2$  to  $LN$ .

- c) Among the  $N$  sets of minimum  $\phi_{\text{eff}}$  data, find the maximum value and note which line was used as the common line for this test. Based on the estimated phase difference criterion, this would be the common line.
- d) At this frequency  $f$ , record the index of this common line in an array.

5. Analytically solve the eigenvalues that will give actual  $e^{\pm \gamma \Delta l}$  values for selected line pairs.

The measurements  $\mathbf{S}^i$  can also be conveniently represented in a cascade matrix  $\mathbf{M}^i$  that transforms the wave variables at Port 2 to the wave variables at Port 1. For a two-port device, the relationship between  $\mathbf{M}$  and  $\mathbf{S}$  is

$$\mathbf{M} = \frac{1}{S_{21}} \begin{bmatrix} (S_{12}S_{21} - S_{11}S_{22}) & S_{11} \\ -S_{22} & 1 \end{bmatrix}, \text{ where } \mathbf{M} \text{ is defined in } \begin{bmatrix} b_1^M \\ a_1^M \end{bmatrix} = \mathbf{M} \begin{bmatrix} a_2^M \\ b_2^M \end{bmatrix}. \quad (12)$$

In actuality, the Multiline method uses switch-term-corrected measurements from pairs of transmission-lines in order to find  $\gamma$ . The Multiline Method paper [1] shows that for any given pair of transmission-line measurements, the eigenvalues of the cascade matrix  $\mathbf{M}^{ij}$  reduce to those of the line pair matrix  $\mathbf{L}^{ij}$  in the absence of noise. Here  $\mathbf{M}^{ij}$  and  $\mathbf{L}^{ij}$  are defined as

$$\mathbf{M}^{ij} \equiv \mathbf{M}^j (\mathbf{M}^i)^{-1} \text{ and} \quad (13)$$

$$\mathbf{L}^{ij} \equiv \mathbf{L}^j (\mathbf{L}^i)^{-1} \equiv \begin{bmatrix} E_1^{ij} & 0 \\ 0 & E_2^{ij} \end{bmatrix} = \begin{bmatrix} e^{-\gamma(\ell_j - \ell_i)} & 0 \\ 0 & e^{+\gamma(\ell_j - \ell_i)} \end{bmatrix}. \quad (14)$$

Solving for the eigenvalues  $\mathbf{M}^{ij}$  then provides an estimate for the propagation-constant of the transmission-line standards. MultiCal uses an analytic expression for the eigenvalues of  $\mathbf{M}^{ij}$ :

$$\lambda_1^{ij}, \lambda_2^{ij} = \frac{1}{2} \left[ (M_{11}^{ij} + M_{22}^{ij}) \pm \sqrt{(M_{11}^{ij} - M_{22}^{ij})^2 + 4M_{12}^{ij}M_{21}^{ij}} \right], \quad (15)$$

where the elements  $M_{mn}$  are given in Eqn. (12), taking as the input  $S$  data the raw switch-term-corrected  $S$ -parameter measurements of Eqn. (8).

At each frequency, these two eigenvalues are computed for the  $N-1$  pairs ( $N$  is the number of transmission-line standards, including the THRU).

*6. Determine correct eigenvalues to  $e^{\pm j\Delta\ell}$  assignment.*

Equation (15) is not yet the end result. We now have two eigenvalues that must be properly assigned to  $E_1^{ij}$  and  $E_2^{ij}$ . At this point, the equations have been developed for transmission-line standards that are identical in every way except for their length, and without consideration of measurement noise (the Multiline Method actually treats the noise as a small perturbation). However, the measurement data used in Eqn. (15) include both electrical noise and contact repeatability noise, plus the effects of small variations in the line standards. The assignment of eigenvalues to appropriate transmission terms may not be directly obvious when attenuation or phase differences between any two standards are small compared to the measurement noise.

To carry out the assignment, MultiCal makes an initial guess:  $E_1^{ij} = \lambda_1^{ij}$  and  $E_2^{ij} = \lambda_2^{ij}$ . It then computes a propagation-constant from the  $E$ 's and compares that to the estimated  $\gamma_{\text{est}}$  defined in Eqns. (9-10). A second guess,  $E_1^{ij} = \lambda_2^{ij}$  and  $E_2^{ij} = \lambda_1^{ij}$  provides another difference between the computed propagation-constant and  $\gamma_{\text{est}}$ . The smaller difference case is generally taken as the

correct answer, though MultiCal applies the following arbitrary tests of significance, especially for high-loss cases, in making a good assignment:

- a) Make starting assignment  $E_1^{ij} = \lambda_1^{ij}$  and  $E_2^{ij} = \lambda_2^{ij}$ .
- b) Form average value

$$E_a = \frac{E_1^{ij} + \frac{1}{E_2^{ij}}}{2}. \quad (16)$$

One possibility is that  $E_a \approx (e^{-\gamma\Delta l} + e^{-(+\gamma\Delta l)})/2 \approx e^{-\gamma\Delta l}$ .

- c) Compute  $\gamma_a\Delta l$  from average  $E_a$ :

$$\gamma_a\Delta\ell = -\ln(E_a) + j2\pi P, \quad (17)$$

where the number of periods  $P$  is needed to estimate the total delay; that is,

$$P = \frac{\text{Im}\{\gamma_{est}\Delta\ell\} - \text{Im}\{-\ln(E_a)\}}{2\pi}, \text{ rounded to the nearest integer.} \quad (18)$$

- d) Compute the relative difference between  $\gamma_a\Delta l$  and  $\gamma_{est}\Delta l$ :

$$D_{a1} = \frac{|\gamma_a\Delta\ell - \gamma_{est}\Delta\ell|}{|\gamma_{est}\Delta\ell|}. \quad (19)$$

- e) Form average value

$$E_b = \frac{E_2^{ij} + \frac{1}{E_1^{ij}}}{2}. \quad (20)$$

The complementary possibility to step b) is that  $E_b \approx (e^{+\gamma\Delta l} + e^{-(-\gamma\Delta l)})/2 \approx e^{+\gamma\Delta l}$ .

- f) Compute the relative difference between  $\gamma_b\Delta l$  and  $-\gamma_{est}\Delta l$ , following Eqns. (17-19):

$$D_{b1} = \frac{|\gamma_b\Delta\ell + \gamma_{est}\Delta\ell|}{|-\gamma_{est}\Delta\ell|}. \quad (21)$$

If the sign of  $\gamma_b\Delta l$  changed from  $\gamma_a\Delta l$ , as might be expected, then  $D_{a1} \approx D_{b1}$ .

- g) Now make the other assignment  $E_1^{ij} = \lambda_2^{ij}$  and  $E_2^{ij} = \lambda_1^{ij}$ .
- h) Repeat steps b) through f), computing a second set of relative differences,  $D_{a2}$  and  $D_{b2}$  for the second assignment.

- i) If  $D_{a1} + D_{b1} < 0.1(D_{a2} + D_{b2})$ , then keep first assignment (much closer fit on first one).
- j) If  $D_{a2} + D_{b2} < 0.1(D_{a1} + D_{b1})$ , then keep second assignment (much closer fit on second one).
- k) If neither case i) nor case j), then check sign of loss term for noise between assignment one and two:

If  $\text{SIGN}(\text{Re}\{\gamma_{a1}\}) \neq \text{SIGN}(\text{Re}\{\gamma_{b1}\})$ , then there's an inconsistency, so make best assignment based on  $D$ 's alone:

$$E_{11}^{ij} = \lambda_{11}^{ij} \text{ and } E_{22}^{ij} = \lambda_{22}^{ij} \text{ if } D_{a1} + D_{b1} < D_{a2} + D_{b2} \text{ and}$$

$$E_{11}^{ij} = \lambda_{22}^{ij} \text{ and } E_{22}^{ij} = \lambda_{11}^{ij} \text{ if } D_{a1} + D_{b1} > D_{a2} + D_{b2}.$$

- l) If neither case i) nor case j), but there is consistency in the signs of the loss terms, make assignment based on attenuation constant (higher-loss case):

If  $|\text{Re}\{\gamma_{a1} - \gamma_{b1}\}| < 0.1|\text{Re}\{\gamma_{a2} + \gamma_{b2}\}|$ , and

$|\text{Re}\{\gamma_{a1}\}|/\text{Im}\{\gamma_{a1}\} > 0.001$  (lossy line case), and

$\text{Re}\{\gamma_{a1}\} > 0$  (a check that it's actually lossy), then assign

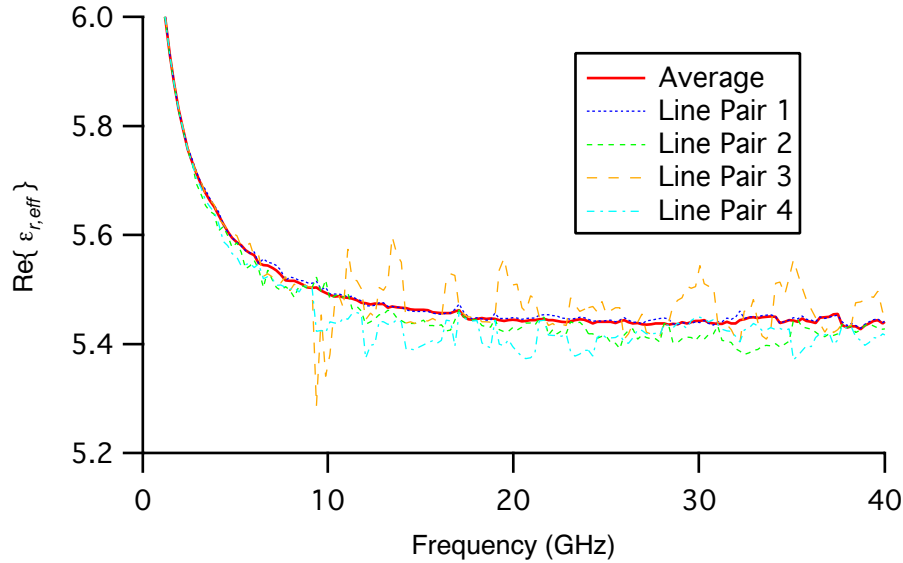
$$E_{11}^{ij} = \lambda_{11}^{ij} \text{ and } E_{22}^{ij} = \lambda_{22}^{ij} \text{ if } D_{a1} + D_{b1} < 0.2 \text{ and}$$

$$E_{11}^{ij} = \lambda_{22}^{ij} \text{ and } E_{22}^{ij} = \lambda_{11}^{ij} \text{ if } D_{a2} + D_{b2} < 0.2.$$

- m) If nothing else works make best assignment based on  $D$ 's alone:

$$E_{11}^{ij} = \lambda_{11}^{ij} \text{ and } E_{22}^{ij} = \lambda_{22}^{ij} \text{ if } D_{a1} + D_{b1} < D_{a2} + D_{b2} \text{ and}$$

$$E_{11}^{ij} = \lambda_{22}^{ij} \text{ and } E_{22}^{ij} = \lambda_{11}^{ij}, \text{ if } D_{a1} + D_{b1} > D_{a2} + D_{b2}.$$



**Figure 3** Example of propagation-constant data computed by MultiCal for thin-film gold coplanar waveguide standards on alumina. The plot shows the phase part of the propagation-constant as the effective relative permittivity, giving the individual computations of four different line pairs, plus MultiCal's estimate (average).

This assignment is made at each frequency point, for each line pair formed with the pre-selected common line. Multiline may make the wrong choice in the presence of noise. Figure 3 displays

the phase constant part of  $\gamma$  as the real part of the effective relative permittivity  $\epsilon'_{r,eff}$  ( $\gamma/2\pi f = j\epsilon_{r,eff}^{1/2}$ ) for line pairs in a set of on-wafer transmission-line standards. For some of the small length differences, we see jumps in the data at certain frequencies due in part to the noise in the assignment process. As this step gives the “observed” value of  $\gamma\Delta l$  for use below, the method treats assignment errors as a random noise process. The effect of the estimation process used in the next step is to place more weight on the observations from the larger length differences.

7. Compute best estimate of  $\gamma$  and the equivalent representation of effective permittivity  $\epsilon_{eff}$ .

Now, MultiCal actually computes the best estimate from among the line pairs using a *linear* least-squares estimator.

After the eigenvalue assignments are made above, MultiCal computes an average  $\gamma\Delta l$  value from an average of  $E_{ij}^{ij}$  and  $1/E_{ij}^{ij}$ , storing the data for each line pair at a given frequency point  $f$ . MultiCal forms a vector  $\mathbf{G}$  of  $N-1$  observations of  $\gamma\Delta l$ . From the user’s line length entries, MultiCal also has a matching vector  $\mathbf{L}$  of  $N-1$  length differences. A simple linear equation relates these with the desired propagation-constant parameter:

$$\mathbf{G} = \gamma\mathbf{L} + \mathbf{e}_r, \quad (22)$$

where  $\mathbf{e}_r$  is the random measurement noise.

In such a system with  $N-1$  linearly independent measurements, we can estimate the parameter  $\gamma$  using a weighted-least-squares method [8]. Minimizing the sum of  $|G_i - \gamma L_i|^2$  over all  $i$ , gives a general formula for an estimate of the propagation-constant  $\gamma$ :

$$\underline{\gamma} = \frac{\mathbf{L}^H \mathbf{W} \mathbf{G}}{\mathbf{L}^H \mathbf{W} \mathbf{L}} \quad (23)$$

where  $\mathbf{L}^H$  designates the Hermitian transpose of  $\mathbf{L}$  for generality, and  $\mathbf{W}$  is a symmetric, positive-definite weighting matrix. This process can be thought of, in part, as multiplying observations of  $(\gamma\Delta l)_i$  by the associated line length difference plus a selected weight, then summing over all  $i$  and dividing by a weighted sum of the length differences squared.

Gauss, and much later Markov, found the optimal weighting matrix to be the inverse of the measurement noise covariance matrix, or  $\mathbf{V}^{-1}$ . This gives a best, linear, unbiased estimator (BLUE) like the one used in the Multiline Method paper [1]:

$$\underline{\gamma} = \frac{\mathbf{L}^H \mathbf{V}^{-1} \mathbf{G}}{\mathbf{L}^H \mathbf{V}^{-1} \mathbf{L}}. \quad (24)$$

In order to ensure stochastically independent observations  $(\gamma\Delta)_i$ , which will be required below to analytically invert a covariance matrix, MultiCal identifies a common line and forms *only* one set of  $N-1$  line pairs.

The elements of the measurement noise covariance matrix are defined generally as

$$\mathbf{V} = \langle \mathbf{e}_r^* \mathbf{e}_r^T \rangle, \quad (25)$$

where  $\mathbf{e}_r^*$  denotes the complex conjugate,  $\mathbf{e}_r^T$  denotes the transpose, and  $\langle \rangle$  denotes the expectation value. If we simply considered the measurement noise values of the observations  $\mathbf{e}_r$  to be mutually independent and identically distributed with zero mean and a variance  $\sigma^2$ , we would obtain the expectation value  $\langle \mathbf{e}_r \rangle = 0$ , and  $\mathbf{V} = \langle \mathbf{e}_r^* \mathbf{e}_r^T \rangle = \sigma^2 \mathbf{I}$ . However, the Multiline method considers the noise differently. It finds  $\mathbf{e}_r$  from the difference of two line measurements, since the observations in  $\mathbf{G}$  can be thought of as the ratio of two line measurements. We can express  $\mathbf{e}_r$  in terms of  $k$ , the noise in individual line measurements:

$$e_i = k_{com} + k_i, \quad (26)$$

where the  $k$ 's are the composite measurement noise for the measurement of the common line and the  $i$ th line in the pair, respectively. Considering a two-observation example for clarity, we find

$$\mathbf{e}_r^* \mathbf{e}_r^T = \begin{bmatrix} (k_{com}^* k_{com} + k_1^* k_1 + k_{com}^* k_1 + k_1^* k_{com}) & (k_{com}^* k_{com} + k_1^* k_2 + k_{com}^* k_2 + k_1^* k_{com}) \\ (k_{com}^* k_{com} + k_2^* k_1 + k_{com}^* k_1 + k_2^* k_{com}) & (k_{com}^* k_{com} + k_2^* k_2 + k_{com}^* k_2 + k_2^* k_{com}) \end{bmatrix}. \quad (27)$$

Assuming no correlation between the mixed terms, and taking the expectation values of each element, our 2x2  $\mathbf{V}$  matrix becomes

$$\mathbf{V} = \begin{bmatrix} 2\sigma_k^2 & \sigma_k^2 \\ \sigma_k^2 & 2\sigma_k^2 \end{bmatrix}, \quad (28)$$

where  $\sigma_k^2$  is the variance of the composite noise in the individual line measurements. As the multiline paper shows, the elements of the inverted matrix  $\mathbf{V}^{-1}$  are given directly by

$$(V^{-1})_{mn} = \left( \delta_{mn} - \frac{1}{N} \right) \frac{1}{\sigma_k^2}, \quad (29)$$

where  $\delta_{mn}$  is the Kronecker delta, and  $N$  still represents the number of line standards; that is, one more than the number of observations. For our example,

$$\mathbf{V}^{-1} = \frac{1}{\sigma_k^2} \begin{bmatrix} 1 - \frac{1}{N} & -\frac{1}{N} \\ -\frac{1}{N} & 1 - \frac{1}{N} \end{bmatrix}. \quad (30)$$

Since  $\sigma_k^2$  shows up in both the denominator and numerator of Eqn. (23), MultiCal does not attempt to determine it in order to solve Eqn. (23); instead it forms  $\sigma_k^2 \mathbf{V}^{-1}$  just knowing  $N$ .

Remember that the ordering of the lines in Step 4 above ensures that MultiCal is not dealing with any other correlation. It is using a minimum set of measurements for  $N-1$  line pairs, so it is not concerned about a nonsingular covariance matrix; consequently the analytic inversion is used directly.

With its computation of  $\gamma$ , MultiCal computes the attenuation constant  $\alpha$ , normalized phase constant  $\beta c/\omega$ , and effective relative permittivity  $\epsilon_{r,eff}$ . These are the data it displays and saves. The relationships between them are given here:

$$\alpha = 20 \text{Log}_{10}(e) \text{Re}\{\gamma\}, \quad (31)$$

$$\beta = \text{Im}\left\{\frac{\gamma}{\omega / 100c}\right\}, \text{ and} \quad (32)$$

$$\epsilon_{r,eff} = -\left[\frac{\gamma}{(\omega / 100c)}\right]^2, \quad (33)$$

where  $e$  is used here as the base of the natural logarithm and  $\omega = 2\pi f$ .

### **Correction Coefficients**

After considering the propagation-constant, we can see how MultiCal determines the correction coefficients of the 8-term model.

In the 8-term model, or error-box formalism, the *measured* cascade matrix of a standard  $\mathbf{M}^i$  is defined in terms of the *actual* cascade matrix  $\mathbf{T}^i$  of a device through two cascade matrices:

$$\mathbf{M}^i = \mathbf{X} \mathbf{T}^i \bar{\mathbf{Y}}, \quad (34)$$

where the overbar denotes that  $\mathbf{Y}$  is the “right-to-left” cascade matrix. The elements of the error boxes can be related to the coefficients in the 8-term model (Fig. 1):

$$\mathbf{X} \equiv R_1 \begin{bmatrix} A_1 & B_1 \\ C_1 & 1 \end{bmatrix} = \frac{1}{e_{10}} \begin{bmatrix} -(e_{00}e_{11} - e_{01}e_{10}) & e_{00} \\ -e_{11} & 1 \end{bmatrix}, \text{ and} \quad (35)$$

$$\bar{\mathbf{Y}} \equiv R_2 \begin{bmatrix} A_2 & C_2 \\ B_2 & 1 \end{bmatrix} = \frac{1}{e_{32}} \begin{bmatrix} -(e_{22}e_{33} - e_{32}e_{23}) & -e_{22} \\ e_{33} & 1 \end{bmatrix}. \quad (36)$$

MultiCal estimates first the two off-axis terms,  $B$  and  $C$  for both ports, then the  $A$  terms and  $R$  coefficients independently. Here, one must solve  $\mathbf{M}^{ij}\mathbf{X} = \mathbf{X}\mathbf{L}^{ij}$  for the Port 1 error box and, by simply exchanging the port indices on the elements of the measured parameters in  $\mathbf{M}^{ij}$ , solves an equation of the same form to get  $\bar{\mathbf{Y}}$ .

In general, TRL formulations are eigenvalue problems with the columns of  $\mathbf{X}$  as the eigenvectors of  $\mathbf{M}^{ij}$  and the diagonal elements of  $\mathbf{L}^{ij}$  as the eigenvalues. The Multiline Method takes advantage of this and normalizes by  $A$  the elements in the first column of the  $\mathbf{X}$ . It then finds  $B_1$  and  $(C/A)_1$  by forming a set of  $N-1$  observations out of the  $\mathbf{M}^{ij}$  matrices for each line pair, using the previously identified common line at each frequency point.

When we solve  $\mathbf{M}^{ij}\mathbf{X} = \mathbf{X}\mathbf{L}^{ij}$  analytically, we have a common  $R_1$  factor on both sides and a common  $(1/S_{21}^j S_{12}^i)$  factor in all elements of the  $\mathbf{M}^{ij}$ . MultiCal solves  $B$  and  $(C/A)$  without knowing  $R_1$  and using a new matrix  $\boldsymbol{\tau}$ , where  $\boldsymbol{\tau} = (S_{21}^j S_{12}^i) \mathbf{M}^{ij}$ . This matrix is found by again inserting the switch-term corrected  $S$ -parameter values into the analytic expressions. The eigenvalues of  $\boldsymbol{\tau}$  are similar to Eqn. (15):

$$\lambda_{\tau 1}^{ij}, \lambda_{\tau 2}^{ij} = \frac{1}{2} \left[ (\tau_{11}^{ij} + \tau_{22}^{ij}) \pm \sqrt{(\tau_{11}^{ij} - \tau_{22}^{ij})^2 + 4\tau_{12}^{ij}\tau_{21}^{ij}} \right]. \quad (37)$$

Using Eqn. (37), we find two cases for  $B_1$  and  $(C/A)_1$  in solving  $\mathbf{M}^{ij}\mathbf{X} = \mathbf{X}\mathbf{L}^{ij}$ :

$$B_1^a = \frac{\tau_{12}^{ij}}{\lambda_{\tau 1}^{ij} - \tau_{11}^{ij}}, \quad (38)$$

$$\left( \frac{C}{A} \right)_1^a = \frac{\tau_{21}^{ij}}{\lambda_{\tau 2}^{ij} - \tau_{22}^{ij}}, \text{ and} \quad (39)$$

$$B_1^b = \frac{\lambda_{\tau 1}^{ij} - \tau_{22}^{ij}}{\tau_{21}^{ij}}, \quad (40)$$



$$\left(\frac{C}{A}\right)_1^b = \frac{\lambda_{\tau_2}^{ij} - \tau_{11}^{ij}}{\tau_{12}^{ij}}. \quad (41)$$

MultiCal now operates on each line pair at each frequency in the list. Since it again might not be obvious which eigenvalue should be associated with which choice of root, there is a duplicity of possibilities in Eqns. (38-41). For example, we could just as well form Eqn. (38) using  $\lambda_{\tau_2}^{ij}$  instead of  $\lambda_{\tau_1}^{ij}$ . For each of the four possible assignments above, MultiCal compares the two cases for  $B$  and  $(C/A)$  to estimates based on MultiCal's  $\gamma$ :

$$B_{est} = \frac{\tau_{12}^{ij}}{e^{+\gamma(l_j - l_i)} - \tau_{11}^{ij}} \text{ and} \quad (42)$$

$$\left(\frac{C}{A}\right)_{est} = \frac{\tau_{21}^{ij}}{e^{-\gamma(l_j - l_i)} - \tau_{22}^{ij}}. \quad (43)$$

The case closest to these estimates becomes the *observation* of  $B$  and  $(C/A)$ .

MultiCal next computes the right-to-left error box for Port 2 in the same manner using Eqns. (37-43), but exchanging the port indices on the  $S$ -parameters used to form the  $\tau^{ij}$ . For example, in solving for  $X$ , MultiCal uses the normal  $\tau_{12}^{ij}$

$$\tau_{12}^{ij} = S_{11}^i (S_{11}^j S_{22}^j - S_{21}^j S_{12}^j) - S_{11}^j (S_{11}^i S_{22}^i - S_{21}^i S_{12}^i); \quad (44)$$

in solving for  $\overline{Y}$ . MultiCal then uses the port-exchanged elements

$$\overline{\tau}_{12}^{ij} = S_{22}^i (S_{22}^j S_{11}^j - S_{12}^j S_{21}^j) - S_{22}^j (S_{22}^i S_{11}^i - S_{12}^i S_{21}^i). \quad (45)$$

MultiCal builds arrays  $B_1$ ,  $(C/A)_1$ ,  $B_2$ , and  $(C/A)_2$  out of the  $N-1$  available observations.

MultiCal estimates values for the  $B$  and  $(C/A)$  parameters following a method similar to Eqn. (24). Here, however, MultiCal does not use an independent variable such as line-length difference, but uses a specific inverted covariance matrix that includes line-length difference information in describing the measurement noise variance. The estimate for either  $B_1$  or  $B_2$  can be written as

$$\underline{B} = \frac{\mathbf{h}^\top \mathbf{V}_B^{-1} \mathbf{B}}{\mathbf{h}^\top \mathbf{V}_B^{-1} \mathbf{h}} = (\mathbf{h}^\top \mathbf{V}_B^{-1} \mathbf{B}) \sigma_B^2, \quad (46)$$

where  $\mathbf{h}$  is a vector with all elements  $h_i=1$ , and the variance  $\sigma_B^2$  is the sum of all the elements in the inverted measurement noise covariance matrix  $\mathbf{V}_B^{-1}$ . This covariance matrix differs from the one used to estimate  $\gamma$  above but is the same for both  $B_1$  and  $B_2$ . Assuming that the noise in the measurements of the individual lines are uncorrelated and that Ports 1 and 2 of the instrument are equally noisy, the Multiline Method paper [1] develops a scaled covariance matrix  $\mathbf{V}_B$ ; that is, one where the measurement noise variance factor is not included (this factor will cancel in Eqn. 46, so it is not determined). In MultiCal, the elements of the scaled  $\mathbf{V}_B$  (neglecting the  $\sigma_k^2$  factor) are found in three parts: for the diagonal elements

$$V_{B,mn}|_{m=n} = \frac{|e^{-\gamma(l_m-l_{com})}|^2 + \frac{1}{|e^{-\gamma(l_m-l_{com})}|^2} + 2\left(|e^{-\gamma l_m}| |e^{-\gamma l_{com}}|\right)^2}{|e^{-\gamma(l_m-l_{com})} - e^{+\gamma(l_m-l_{com})}|^2}, \quad (47)$$

for the upper triangle elements

$$V_{B,mn}|_{m<n} = \frac{e^{-\gamma(l_m-l_{com})} \left(e^{-\gamma(l_m-l_{com})}\right)^* + |e^{-\gamma l_{com}}|^2 e^{-\gamma l_m} \left(e^{-\gamma l_n}\right)^*}{\left(e^{-\gamma(l_m-l_{com})} - e^{+\gamma(l_m-l_{com})}\right) \left(e^{-\gamma(l_n-l_{com})} - e^{+\gamma(l_n-l_{com})}\right)^*}, \quad (48)$$

and for the lower triangle elements

$$V_{B,mn}|_{m>n} = \left(V_{B,nm}\right)^*. \quad (49)$$

Here,  $\gamma$  is the accurate propagation-constant estimate that MultiCal computed above,  $l_{com}$  is the length of the common line identified before at the given frequency, and  $l_m$  or  $l_n$  are the lengths of the other lines in the pairs.

For the first time, MultiCal uses a numerical inversion to compute  $\mathbf{V}_B^{-1}$ , and uses this to compute the estimate in Eqn. (46). While we might not arrive at the ultimate minimum variance, the line ordering procedure still provides a good estimate that is unbiased under the stated assumptions.

MultiCal estimates the  $C/A$  parameters in a similar manner using a different covariance matrix, again scaled to remove the measurement noise variance factor:

$$\underline{C/A} = \frac{\mathbf{h}^T \mathbf{V}_C^{-1} \mathbf{C}}{\mathbf{h}^T \mathbf{V}_C^{-1} \mathbf{h}} = \left(\mathbf{h}^T \mathbf{V}_C^{-1} \mathbf{C}\right) \sigma_C^2, \quad (50)$$

wherem for the diagonal elements of  $\mathbf{V}_C$

$$V_{C,mn}|_{m=n} = \frac{\left| e^{-\gamma(l_m - l_{com})} \right|^2 + \frac{1}{\left| e^{-\gamma(l_m - l_{com})} \right|^2} + \frac{2}{\left( \left| e^{-\gamma l_{com}} \right| \left| e^{-\gamma l_m} \right| \right)^2}}{\left| e^{-\gamma(l_m - l_{com})} - e^{+\gamma(l_m - l_{com})} \right|^2}, \quad (51)$$

for the upper triangle elements

$$V_{C,mn}|_{m < n} = \frac{\frac{1}{e^{-\gamma(l_m - l_{com})} \left( e^{-\gamma(l_n - l_{com})} \right)^*} + \frac{1}{\left| e^{-\gamma l_{com}} \right|^2 e^{-\gamma l_m} \left( e^{-\gamma l_n} \right)^*}}{\left( e^{-\gamma(l_m - l_{com})} - e^{+\gamma(l_m - l_{com})} \right) \left( e^{-\gamma(l_n - l_{com})} - e^{+\gamma(l_n - l_{com})} \right)^*}, \quad (52)$$

and for the lower triangle elements

$$V_{C,mn}|_{m > n} = \left( V_{C,nm} \right)^*. \quad (53)$$

Users of MultiCal may be familiar with the Normalized Standard Deviation plot the program can supply. These data simply represent the arithmetic average  $(\sigma_B + \sigma_C)/2$ , using the terms from Eqns. (46 & 50).

In the last steps of this section, MultiCal computes estimates for the  $A$  and  $R$  parameters. The Multiline Method paper did not document their solution before, but they follow from earlier TRL solutions.

To find the  $A$ 's, MultiCal computes an estimated value for the reflection coefficient of all reflect standards. In practice, MultiCal can use multiple reflection standards to find an average  $A$ , but for the purpose of this document, we will consider only a single REFLECT standard. The user can select whether the reflection standard is an open or short circuit, and can specify the location of the reflection plane relative to the connector plane (probe-tips). MultiCal can transform this frequency-flat reflection coefficient if the reflection plane is not located where the middle of the THRU standard would be placed; that is,

$$\Gamma_{r,est} = \Gamma_{refl} e^{-2\gamma(\ell_{refl} - \ell_{THRU/2})}, \quad (54)$$

where  $\Gamma_{refl}$  is the reflection coefficient of the standard provided by the users,  $\ell_{refl}$  is the length entry for the standard specified by the user (taking symmetric reflection standards), and  $\ell_{THRU/2}$  gives the location of the calibration reference place as one half of the THRU length.

MultiCal then determines the  $A$ 's at each frequency. It finds the product  $A_p = A_1 A_2$  from measurements of the THRU and the ratio  $A_r = A_1 / A_2$  from measurements of the REFLECT. With those two values, it finds the individual coefficients. In the process,  $\Gamma_{refl}$  will be used only to find the sign when taking the square root of  $A_p$ .

If we consider the THRU to be the connection of the two ports at the measurement reference plane, we define an ideal connection with total reciprocal transmission and zero reflection. Solving for

$$\mathbf{M}^{THRU} = \mathbf{X} \mathbf{I} \bar{\mathbf{Y}} \quad (55)$$

gives the product of  $A_1 A_2$  as

$$A_p = - \frac{B_1 B_2 - B_1 S_{22}^{THRU} - B_2 S_{11}^{THRU} + (S_{11} S_{22} - S_{21} S_{12})^{THRU}}{1 - \left(\frac{C}{A}\right)_1 S_{11}^{THRU} - \left(\frac{C}{A}\right)_2 S_{22}^{THRU} + \left(\frac{C}{A}\right)_1 \left(\frac{C}{A}\right)_2 (S_{11} S_{22} - S_{21} S_{12})^{THRU}}. \quad (56)$$

Other solutions for the product can be found, but this one eliminates factors with potentially small denominators. To determine Eqn. (56), four analytic expressions can be formed with Eqn. (55) relating the elements in the cascade measurement matrix with the error-box coefficients. Taking the equation with an  $A_1 A_2$  term and dividing by each of the other three equations isolates a desired  $1/A_1 A_2$  term. In order to get rid of troublesome factors such as  $1/S_{11}^{THRU}$ , certain equations must be multiplied by a  $\pm(S_{11}^{THRU} S_{22}^{THRU})$  factor before summing the three remaining equations and solving for  $A_p$ .

To find  $A_r$ , we consider the isolated error boxes with the reflection standards connected. Taking the reflections at each port to be identical ( $\Gamma_1 = \Gamma_2$ ) and using a flow-diagram solution gives a set of equations where

$$A_r = \frac{A_1 \Gamma_1}{A_2 \Gamma_2} = \frac{S_{11}^{Re fl} - B_1}{1 - S_{11}^{Re fl} (C/A)_1} \frac{1 - S_{22}^{Re fl} (C/A)_2}{S_{22}^{Re fl} - B_2}. \quad (57)$$

Here, the  $S_{mn}^{refl}$  are the measurements of the reflection standard. Finding  $A_1^2 = A_p A_r$  leaves a sign ambiguity. To find the sign of  $A_1$ , MultiCal subtracts two normalized vectors, then tests the magnitude of the difference vector:

$$\left| \frac{\Gamma_{r,est}}{|\Gamma_{r,est}|} - \frac{\Gamma_{trial}}{|\Gamma_{trial}|} \right| > \sqrt{2}, \quad (58)$$

where  $\Gamma_{trial}$  is the reflection coefficient found in Port 1 solution as

$$\Gamma_{trial} = \frac{S_{11}^{Re\ fl} - B_1}{\sqrt{A_p A_r (1 - S_{11}^{Re\ fl} (C/A)_1)}}. \quad (59)$$

If Eqn. (58) is TRUE,  $\Gamma_{trial}$  will have a component opposite to that of  $\Gamma_{r,est}$  and

$$A_1 = -\sqrt{A_p A_r}; \quad (60)$$

otherwise

$$A_1 = \sqrt{A_p A_r}. \quad (61)$$

The remaining assignment is simply  $A_2 = A_1/A_r$ .

In our formulation, there are two sets of *ABCR* parameters, one for each port, but the strict 8-term model has only seven independent coefficients, with one common prefactor equivalent to our  $R_1 R_2$ . Following much the same procedure in finding the  $A_p$  from the measurement of the THRU, we can find a solution of the product  $R_1 R_2$ . Various methods of assigning the  $R$ s have been described [4,9] before. The MultiCal code divides the contributions for each port based on the measured transmission parameters of a reciprocal THRU standard:

$$R_1 = \frac{S_{12}^{THRU}}{1 - \left(\frac{C}{A}\right)_1 S_{11}^{THRU} - \left(\frac{C}{A}\right)_2 S_{22}^{THRU} + \left(\frac{C}{A}\right)_1 \left(\frac{C}{A}\right)_2 (S_{11} S_{22} - S_{21} S_{12})^{THRU}} \text{ and} \quad (62)$$

$$R_2 = \frac{S_{21}^{THRU}}{1 - \left(\frac{C}{A}\right)_1 S_{11}^{THRU} - \left(\frac{C}{A}\right)_2 S_{22}^{THRU} + \left(\frac{C}{A}\right)_1 \left(\frac{C}{A}\right)_2 (S_{11} S_{22} - S_{21} S_{12})^{THRU}}. \quad (63)$$

From this, MultiCal has estimates of the propagation-constant  $\gamma$  and the 8-term VNA model as two sets of *ABCR* parameters. The correction coefficients are defined for a calibration reference plane positioned  $l_{THRU}/2$  away from the connection, or probe-tip plane, and with a reference impedance equal to the characteristic impedance of the line standards,  $Z_{ref} = Z_0$  (ideal match conditions at the reference plane).

### **Reference Plane and Impedance Transformations**

If the user so desires, the reference plane can be located in a position along a length of uniform transmission-line identical to that of the transmission-line standards. MultiCal transforms the  $A$  and the  $R$  terms to create a new set of correction coefficients that compensate for this translation:

$$A_i^\Delta = A_i e^{-2\gamma\Delta\ell} \text{ and} \quad (64)$$

$$R_i^\Delta = R_i e^{-2\gamma\Delta\ell}, \quad (65)$$

where  $\Delta\ell$  is the user-specified change in reference plane position (physical length of the transmission-line).

Additionally, the user can transform  $Z_{ref}$  to another more desirable connection impedance. With  $Z_{ref}$  and  $Z_0$  known, MultiCal can modify its coefficients. The user can supply a file that specifies the characteristic impedance of the transmission-line standards at each frequency point in the list, or the user can specify one transmission-line parameter  $C_0$ , in units of pF/10<sup>-2</sup> m and let MultiCal estimate the  $Z_0$ . The latter case works for lines of negligible conductance, where the line losses are dominated by the conductor resistance. Here, MultiCal estimates  $Z_0$  as

$$Z_0 \approx \sqrt{\frac{\epsilon_{r,eff}}{c C_0}}, \quad (66)$$

where  $\epsilon_{r,eff}$  is the result of MultiCal's propagation-constant solution and  $c$  is the speed of light.

Using either a  $Z_0$  file or Eqn. (66), we can find the two-port impedance transformation matrix. MultiCal cascades two matrices to find new  $ABC$  parameters for both Port 1 and 2 (The  $A$  and  $R$  inputs used here may in fact be the reference plane corrected parameters  $A^\Delta$  and  $R^\Delta$  from above):

$$\mathbf{T}_i^Z = \begin{bmatrix} A_i & B_i \\ A_i(C/A)_i & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{1-\Gamma_Z^2}} & \frac{-\Gamma_Z}{\sqrt{1-\Gamma_Z^2}} \\ \frac{-\Gamma_Z}{\sqrt{1-\Gamma_Z^2}} & \frac{1}{\sqrt{1-\Gamma_Z^2}} \end{bmatrix}, \quad (67)$$

where

$$\Gamma_Z = \frac{Z_0 - Z_{ref}}{Z_0 + Z_{ref}}. \quad (68)$$

The elements of  $\mathbf{Z}_i$  give the new values of the  $ABCR$  parameters as

$$A_i^Z = \frac{T_{i,11}^Z}{T_{i,22}^Z}, \quad (69)$$

$$B_i^Z = \frac{T_{i,12}^Z}{T_{i,22}^Z}, \text{ and} \quad (70)$$

$$\left(\frac{C}{A}\right)_i^Z = \frac{T_{i,21}^Z}{T_{i,11}^Z}. \quad (71)$$

MultiCal solves the impedance transformed  $R$ 's as

$$R_1^Z = R_1 \left( 1 - \frac{B_2(C/A)_2}{A_1} \right) \left( \frac{A_1^Z}{1 - B_2^Z(C/A)_2^Z} \right) \frac{T_{1,22}^Z}{T_{2,22}^Z} \text{ and} \quad (72)$$

$$R_2^Z = R_2 \left( 1 - \frac{B_1(C/A)_1}{A_2} \right) \left( \frac{A_2^Z}{1 - B_1^Z(C/A)_1^Z} \right) \frac{T_{2,22}^Z}{T_{1,22}^Z}. \quad (73)$$

### **Measurement Correction**

Now, we have all the required elements for the error boxes, including reference plane and reference impedance transformations. MultiCal uses the transformed  $ABCR$  parameters to analytically form the  $X^{-1}$  and  $Y^{-1}$ -overbar correction matrices. It corrects a device-under-test (DUT) measurement to produce the actual cascade matrix  $T$ , from which MultiCal extracts the  $S$ -parameters:

$$T^{DUT} = X^{-1} M^{DUT} \overline{Y}^{-1}. \quad (74)$$

If MultiCal is controlling an automated VNA, it can also form the 12 terms used in the common VNA correction model from its 8  $ABCR$  terms [4]. The assignments are shown here using the definitions of Fig. 1:

$$e_{00} = B_1, \quad (75)$$

$$e_{11} = -\left(\frac{C}{A}\right)_1 A_1, \quad (76)$$

$$e_{10}e_{01} = A_1 - B_1 \left(\frac{C}{A}\right)_1 A_1, \quad (77)$$

$$e_{30} = Iso_F, \quad (78)$$

$$e_{22} = \frac{A_2(\Gamma_R - (C/A)_2)}{1 - B_2\Gamma_R}, \quad (79)$$

$$e_{10}e_{32} = \frac{A_1A_2}{R_1(1 - B_2\Gamma_R)}, \quad (80)$$

$$e'_{33} = B_2, \quad (81)$$

$$e'_{22} = -\left(\frac{C}{A}\right)_2 A_2, \quad (82)$$

$$e'_{23}e'_{32} = A_2 - B_2\left(\frac{C}{A}\right)_2 A_2, \quad (83)$$

$$e'_{03} = Iso_R, \quad (84)$$

$$e'_{11} = \frac{A_1(\Gamma_F - (C/A)_1)}{1 - B_1\Gamma_F}, \text{ and} \quad (85)$$

$$e'_{23}e'_{01} = \frac{A_1A_2}{R_2(1 - B_1\Gamma_F)}. \quad (86)$$

The VNA can then apply its own correction to DUT measurements in real time using MultiCal's calculations.

## Summary

We have documented here methods used in the NIST MultiCal<sup>®</sup> program that have not been presented before. We showed how this implementation of the Multiline TRL calibration is not simply a solution of simultaneous error correction equations, but rather an evolved set of tests and traps, taking advantage of analytic formulations that can branch to specific solutions that are numerically well behaved. We showed how MultiCal applies a linear-least-squares estimator in finding an accurate estimate of the transmission-line propagation-constant, as well the 8 terms in the error-box formulation. We hope this document will assist the authors of other Multiline TRL algorithms, possibly serving as a launching point for new discussions of this approach.



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